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Isotropization and thermalization in heavy-ion collisions

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Abstract. We argue that isotropization and, consequently, thermalization of the system of gluons and quarks produced in an ultrarelativistic heavy-ion collision does not follow from Feynman diagram analysis to any order in the coupling constant. We conclude that the apparent thermalization of quarks and gluons, leading to success of perfect fluid hydrodynamics in describing heavy-ion collisions at RHIC, can only be attributed to the non-perturbative QCD effects not captured by Feynman diagrams. We proceed by modeling these non-pertrubative thermalization effects using viscous hydrodynamics. We point out that matching Color Glass Condensate inital conditions with viscous hydrodynamics leads to a continuous evolution of all the components of the energy-momentum tensor and, unlike the case of ideal hydrodynamics, does not give rise to a discontinuity in the longitudinal pressure. An important consequence of such a matching is a relationship between the thermalization time and shear viscosity: we observe that small viscosity leads to short thermalization time.

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1 Introduction: isotropization versus free streaming

The results presented here are mainly based on the work done in [1,2], but also include new developments described in sect. 4.

Similar to the original Bjorken hydrodynamics approach [3], let us consider a central high-energy collision of two very large nuclei. For simplicity, here we will discuss the case where the distribution of particles is independent of space-time rapidity $\eta = (1/2)\ln(x_+/x_-)$, where $x_\pm = (t \pm z)/\sqrt{2}$. Since the nuclei are very large the transverse coordinate dependence can also be neglected for most physical quantities, leaving only the dependence on the proper time $\tau = \sqrt{2x_+x_-}$. For this geometry, one can show that the most general energy-momentum tensor can be written as (at $z = 0$) [1]

$$T^{\mu\nu} = \begin{pmatrix} \epsilon(\tau) & 0 & 0 & 0 \\ 0 & p(\tau) & 0 & 0 \\ 0 & 0 & p(\tau) & 0 \\ 0 & 0 & 0 & p_3(\tau) \end{pmatrix}, \tag{1}$$

where the z-axis is taken along the beam direction, and x, y-axes are in the transverse direction. Applying the conservation of energy-momentum tensor condition

$$\partial_\mu T^{\mu\nu} = 0 \tag{2}$$

to the energy-momentum tensor that gives eq. (1) at $z = 0$ we obtain

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau} = -\frac{\epsilon + p_3}{\tau}. \tag{3}$$

There are two interesting cases one can consider:

i) if $p_3 = 0$ longitudinal pressure vanishes and, due to eq. (3), we get

$$\epsilon \sim \frac{1}{\tau}, \tag{4}$$

such that the total energy $E \approx \epsilon\,\tau =$ const. This case is known as *free streaming*: the system expands freely without loosing any energy.

ii) if $p_3 = p$ the energy-momentum tensor in eq. (1) becomes *isotropic*. This is the case of ideal Bjorken hydrodynamics [3]. Equation (3) with $p_3 = p$ was derived in [3]. If combined with the ideal gas equation of state, $\epsilon = 3\,p$, it gives

$$\epsilon \sim \frac{1}{\tau^{4/3}} \tag{5}$$

or, for other equations of state,

$$\epsilon \sim \frac{1}{\tau^{1+\Delta}} \qquad \text{with} \qquad \Delta > 0. \tag{6}$$

Equation (3) demonstrates that changes in the total energy $E \approx \epsilon\,\tau$ (or, equivalently, deviations from $\epsilon \sim 1/\tau$ scaling) are due to work done by the longitudinal pressure p_3. The classical initial conditions in the Color Glass

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Condensate (CGC) approach [4] yield the free-streaming final state with $p_3 = 0$. A thermalized quark-gluon plasma (QGP) is characterized by non-zero p_3, leading to the energy density scaling as shown in eq. (6). Therefore, below we will understand *isotropization*, which is the necessary condition for *thermalization*, as dynamical generation of non-zero longitudinal pressure $p_3 \neq 0$, or, equivalently, deviations from the scaling of eq. (4) leading to the scaling of eq. (6).

2 Formal argument

An extensive search of the diagrams which would bring in the desired deviations from the scaling of eq. (4) carried out by the author did not yield any positive results: while many diagrams have contributions to ϵ scaling as shown in eq. (6), such terms are *always* subleading additive corrections to the leading (at late times) terms scaling as shown in eq. (4). In fact one can construct an argument [1] demonstrating that the leading contribution to energy density from any-order diagrams scales as $\epsilon \sim 1/\tau$. The argument is presented below.

We begin by considering a gluon field generated by an arbitrary Feynman diagram [1], illustrated in fig. 1. In $\partial_\mu A^\mu = 0$ covariant gauge the gluon field can be written as

$$A^a_\mu(x) = -i \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{k^2 + i\epsilon k_0} J^a_\mu(k), \tag{7}$$

where the function $J^a_\mu(k)$ denotes the rest of the diagram in fig. 1 (the truncated part), which depends on the momenta of other outgoing gluons as well. Indeed the gluon field can be defined as a simple function only in the classical case: the "field" in eq. (7) should be thought of as a Feynman diagram in fig. 1 with one of the outgoing gluon lines being off mass-shell, *i.e.*, a generalization of the classical field which we will need in calculating energy density [1]. (The expression in eq. (7) can also be thought of as an operator equation.)

Substituting eq. (7) into the expression for energy-momentum tensor

$$T^{\mu\nu} = \left\langle -F^{a\,\mu\rho}\, F^{a\,\nu}{}_{\rho} + \frac{1}{4}\, g^{\mu\nu}\, (F^a_{\rho\sigma})^2 \right\rangle, \tag{8}$$

averaging over the nuclear wave functions and employing the symmetries of the collision of two identical nuclei, we obtain the energy density due to the gluon field [1]:

$$\epsilon = \int \frac{\mathrm{d}^4 k\, \mathrm{d}^4 k'}{(2\pi)^8} \frac{e^{-ik\cdot x - ik'\cdot x}}{(k^2 + i\epsilon k_0)(k'^2 + i\epsilon k'_0)} \times \left\{ \frac{1}{2} \left[\left(\frac{\tau}{x_+} \right)^2 k_+ k'_+ - \underline{k}^2 \right] f_1(k^2, k'^2, k\cdot k', k_T) + \ldots \right\}, \tag{9}$$

where $f_1(k^2, k'^2, k\cdot k', k_T)$ is some unknown function (a "form-factor") and the ellipsis indicate addition of two more similar terms with different "form-factors" f_2 and f_3.

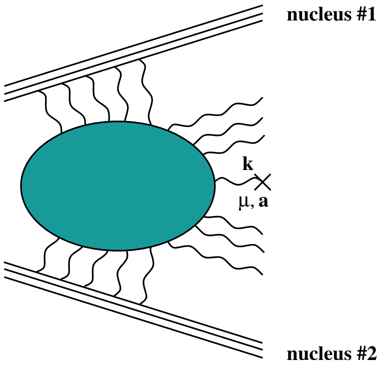

Fig. 1. Gluon "field" generated by an arbitrary-order diagram (see text).

Rewriting each "form-factor" as

$$f_i(k^2, k'^2, k\cdot k', k_T) = f_i(k^2 = 0, k'^2 = 0, k\cdot k' = 0, k_T) + \left[f_i(k^2, k'^2, k\cdot k', k_T) - f_i(k^2 = 0, k'^2 = 0, k\cdot k' = 0, k_T) \right] \tag{10}$$

and using the fact that the square of the truncated part of the diagram gives a cross-section

$$\frac{\mathrm{d}N}{\mathrm{d}^2 k\, \mathrm{d}y} = \frac{1}{2(2\pi)^3} \left\langle \left\langle J^{a\,\rho}(k)\, J^a_\rho(-k) \right\rangle \right\rangle \Big|_{k^2=0} \tag{11}$$

we conclude that, keeping only the first term on the right-hand side of eq. (10) for all "form-factors" in eq. (9) yields

$$\epsilon = \frac{\pi}{2} \int \mathrm{d}^2 k \frac{\mathrm{d}N}{\mathrm{d}^2 k\, \mathrm{d}\eta\, \mathrm{d}^2 b}\, k_T^2 \left\{ [J_1(k_T\tau)]^2 + [J_0(k_T\tau)]^2 \right\} \approx \frac{1}{\tau} \int \mathrm{d}^2 k \frac{\mathrm{d}N}{\mathrm{d}^2 k\, \mathrm{d}\eta\, \mathrm{d}^2 b}\, k_T, \tag{12}$$

where the last equality is valid for late proper times τ. Since, as was shown in [1], each factor of k^2, k'^2 or $k\cdot k'$ gives a factor of $1/\tau$, the terms in the square brackets of eq. (10) give a subleading (compared to eq. (12)) contribution to energy density at late times τ and can be safely neglected[1]. We have shown that any diagram and/or any set of diagrams contributing to gluon production cross-section lead to energy density scaling as in eq. (4), *i.e.*, that isotropization and, consequently, thermalization do not take place in perturbation theory analysis of the collisions.

The main assumption of the argument presented above is the existence of multiplicity of produced gluons $\mathrm{d}N/\mathrm{d}^2 k\, \mathrm{d}y$, which is the essential assumption of QCD perturbation theory. This is what makes our argument perturbative.

3 Physical argument

Now let us present a physical argument demonstrating the origin of the power of 4/3 in eq. (5) and explaining why

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[1] We assume that f_i's are continuous functions of their arguments.

it is impossible to be achieved in perturbation theory [2]. Let us assume that thermalization does take place at some time τ_{th}. If a gauge invariant time τ_{th} exists, we can put the QCD coupling constant $g = 0$ for all times $\tau > \tau_{th}$ without violating gauge invariance. Bjorken hydrodynamics in the $g = 0$ limit is governed by the ideal gas equation of state $\epsilon = 3\,p$, which leads to the energy density scaling as shown in eq. (5). (For small but non-zero g, eq. (5) would get an $o(g^2)$ negative correction to 4/3: the expansion in g would still be around the power of 4/3.) Due to eq. (3), the scaling of eq. (5) in the $g = 0$ limit of Bjorken hydrodynamics means that $p_3 \neq 0$ and the gas of *non-interacting* particles is *doing work* in the longitudinal direction! What causes such a behavior of the system? The problem lies in the ideal gas equation of state, $\epsilon = 3\,p$, which assumes that the ideal gas is in contact with some *external thermal bath*. Such external thermal bath could be a background field or a box containing the gas: the ideal gas of non-interacting particles stays thermal through the interactions between the gas particles and the thermal bath. This is the only interaction allowed in the $g \to 0$ limit and it is responsible for the work done by the non-interacting gas. Since there is no such external thermal bath in heavy-ion collisions, the scaling of eq. (5) is impossible to be achieved at small coupling.

Without the external thermal bath the particles in the gas would be just free streaming, giving the physically correct energy density scaling of eq. (4). Of course, at a fixed time τ hydro is not applicable in the $g = 0$ limit, since the mean free path of the particles would exceed the longitudinal size of the system. However, *if* thermalization does happen, for any fixed arbitrary small g, if we wait long enough hydrodynamics should become applicable, leading to the scaling arbitrary close to that of eq. (5) and doing work in the longitudinal direction which would be mostly due to contact with the non-existing external thermal bath. Therefore, we arrive at a contradiction, demonstrating that hydrodynamics is not achievable at small coupling. At large coupling, non-perturbative effects may mimic the external thermal bath, possibly leading to the energy density scaling shown in eq. (6).

4 Matching color glass initial conditions and viscous hydrodynamics

If thermalization does take place via some non-perturbative mechanism, understanding its dynamics would be extremely hard. Instead, let us ask a pragmatic question: what is expected from a successful thermalization scenario? Apart from giving us correct dynamical mechanism for the generation of longitudinal pressure and the onset of isotropization, one may expect a thermalization scenario to yield us an estimate of the time when energy-momentum tensor would become symmetric, $T_{\mu\nu} = \text{diag}\{\epsilon, p, p, p\}$ at $z = 0$ (again we are considering a central collision of two heavy ions in the rapidity-independent approximation for produced particles). At this time ideal hydrodynamics would be initiated and it could be used to describe the subsequent evolution of the system. Unfortunately, as was argued in [5], ideal hydrodynamics is probably unachievable in nature, since the shear viscosity η never completely vanishes[2]. Therefore, one has to use viscous hydrodynamics with the energy-momentum tensor given by

$$\begin{aligned} T_{\mu\nu} &= (\epsilon + p)\, u_\mu\, u_\nu - p\, g_{\mu\nu} \\ &\quad + \eta \left(\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}\, \Delta_{\mu\nu}\, \nabla_\rho\, u^\rho \right), \end{aligned} \tag{13}$$

where u_μ is the velocity profile, $\nabla_\mu \equiv (g_{\mu\nu} - u_\mu\, u_\nu)\, \partial^\nu$ and $\Delta_{\mu\nu} \equiv g_{\mu\nu} - u_\mu\, u_\nu$. In the boost-invariant case eq. (13) leads to

$$T^{\mu\nu} = \begin{pmatrix} \epsilon(\tau) & 0 & 0 & 0 \\ 0 & p(\tau) + \frac{2}{3}\frac{\eta}{\tau} & 0 & 0 \\ 0 & 0 & p(\tau) + \frac{2}{3}\frac{\eta}{\tau} & 0 \\ 0 & 0 & 0 & p(\tau) - \frac{4}{3}\frac{\eta}{\tau} \end{pmatrix}. \tag{14}$$

One can see that viscosity corrections increase transverse pressure and decrease longitudinal pressure in the system.

Now, if we follow [5] and assume that ideal hydrodynamics is unachievable, we can try to address once again the question of what to expect from a thermalization scenario. Our earlier answer would not work anymore: thermalization cannot give us a completely isotropic energy-momentum tensor, since such a tensor is impossible due to viscosity corrections. Therefore, thermalization dynamics can only bring the system to a somewhat isotropic state achieved at certain thermalization time, after which viscous hydrodynamics takes over.

However, the advantage of viscous hydrodynamics energy-momentum tensor in eq. (14) is that it can be *continuously* mapped onto the energy-momentum tensor of color glass initial conditions without including any additional thermalization dynamics. In the rapidity-independent case the energy-momentum tensor due to the classical fields in the color glass is given by [4,1]

$$T^{\mu\nu} = \text{diag}\{\epsilon'(\tau), p'(\tau), p'(\tau), 0\}. \tag{15}$$

We mark the energy density and pressure in eq. (15) with a prime to distinguish them from the appropriate components of the energy-momentum tensor in eq. (14).

Indeed matching the energy-momentum tensors in eqs. (15) and (14) is a rather crude approximation, which, by omitting thermalization dynamics, would only give us a lower bound on thermalization time. Alternatively, one can view viscous hydrodynamics (14) as a model for the non-perturbative thermalization effects: then, matching eqs. (15) and (14) would generate a non-perturbative thermalization scenario.

Requiring that at some "thermalization" time τ_0 the components of energy-momentum tensors in eqs. (15)

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[2] Following the standard convention we will use η to denote shear viscosity in this section, which should not be confused with space-time rapidity η used in the previous sections.

and (14) are equal yields the following set of equations[3]:

$$\begin{cases} \epsilon(\tau_0) = \epsilon'(\tau_0), \\ p(\tau_0) + \frac{2}{3}\,\frac{\eta(\tau_0)}{\tau_0} = p'(\tau_0), \\ p(\tau_0) - \frac{4}{3}\,\frac{\eta(\tau_0)}{\tau_0} = 0. \end{cases} \tag{16}$$

The energy density ϵ' and transverse pressure p' of the CGC initial conditions determine, through eqs. (16), the initial values of energy density, pressure and viscosity of the QGP, along with the matching time τ_0. Indeed, the exact knowledge of QGP thermodynamics would, in principle, allow one to express ϵ, p and η as functions of the system's temperature T. Then eqs. (16) would have only two unknowns —matching time τ_0 and the corresponding temperature at the matching T_0, and would be over-constrained. However, in the case when $T^\mu_\mu = 0$ on both sides of the matching, eqs. (16) have only two independent equations

$$\begin{cases} \epsilon(\tau_0) \approx \epsilon'(\tau_0), \\ p(\tau_0) \approx \frac{4}{3}\,\frac{\eta(\tau_0)}{\tau_0}, \end{cases} \tag{17}$$

and are not over-constrained anymore. The tracelessness condition, $T^\mu_\mu = 0$, is valid in CGC both at the classical level, and at the level of leading logarithmic small-x evolution. On QGP side $T^\mu_\mu = 0$ is valid at very high temperatures, and also $\epsilon = 3\,p$ appears to be a good approximation for the equation of state in the strong-coupling regime. We conclude that solution of eqs. (17) in terms of τ_0 and T_0 is likely to satisfy the third equation in (16) as well.

It is impossible to solve eqs. (17) without knowing the exact dependence of ϵ and p on temperature. Instead, we will find an approximate solution by writing

$$\epsilon(T) = 3\,p(T) = \frac{\pi^2}{30}\,n(g)\,T^4, \tag{18}$$

where $n(g)$ is some function of QCD coupling constant $g(T)$, which, at high temperatures, corresponding to small couplings, just counts the number of quark and gluon degrees of freedom. Following [5] we write for the shear viscosity

$$\eta = f(g^2\,N_c)\,N_c^2\,T^3 \tag{19}$$

with $f(g^2\,N_c)$ some function of the coupling. For the color glass energy density we write

$$\epsilon' = c_E\,\frac{C_F\,Q_s^3}{\alpha_s\,\tau\,2\pi^2}\,, \tag{20}$$

where c_E is some order 1 coefficient to be determined by explicit calculations and Q_s is the saturation scale. Substituting eqs. (18), (19) and (20) into eqs. (17), and neglecting, perhaps unreasonably, the dependence of the coupling constant $g(T)$ on the temperature T, we obtain the matching time

$$\tau_0 \approx \frac{\alpha_s^{1/3}\,[f(g^2\,N_c)\,N_c^2]^{4/3}}{Q_s\,n(g)}\,\frac{120}{\pi^2}\left(\frac{8\,\pi^2}{c_E\,C_F}\right)^{1/3}, \tag{21}$$

which is a good estimate of thermalization time being the only relevant time scale.

Let us analyze the thermalization time from eq. (21). First of all, while, as we argued in the previous sections, perturbative thermalization is dynamically impossible, we can still explore the small-coupling asymptotics of eq. (21) to construct a *lower* bound on perturbative thermalization time. To that end we note that at small values of the coupling $n(g)$ approaches a constant, while $f(g^2\,N_c) \sim 1/g^4$. Using this in eq. (21), and dropping all the constant factors, gives

$$\tau_0 \geq \frac{1}{\alpha_s^{7/3}\,Q_s}, \tag{22}$$

which is precisely the lower bound advocated by Arnold and Lenaghan in [6].

In general, using eq. (19) in eq. (21) to replace $f(g^2\,N_c)\,N_c^2$ with η/T^3 and dropping the numerical factors we conclude that

$$\tau_0 \propto \frac{\alpha_s^{1/3}}{Q_s}\,\left(\frac{\eta}{T^3}\right)^{4/3}. \tag{23}$$

This is the central result of this section. It appears that the matching procedure between the CGC initial conditions and viscous hydrodynamics leads to dependence of thermalization time on shear viscosity. Moreover, the dependence shown in eq. (23) implies that *lower viscosity leads to shorter equilibration time*! This conclusion agrees well with the fact that hydrodynamic simulations demand small shear viscosity and early thermalization time to describe RHIC data. It appears that, if our matching procedure captures correctly at least some features of the actual non-perturbative thermalization dynamics, short thermalization time and small viscosity of the quark-gluon plasma may be related.

Finally, assuming that 't Hooft coupling $g^2 N_c$ is large at RHIC, one can use the result of [5] that $f(g^2\,N_c) = \pi/8$ in the strong-coupling limit (indeed, of $\mathcal{N} = 4$ SYM theory), to estimate the thermalization time τ_0 using eq. (21). Using $\alpha_s = 0.3$, $c_E = 1$, $Q_s = 1.4\,\mathrm{GeV}$ for central RHIC collisions, and $n(g) = 47.5$ for three flavors yields $\tau_0 \approx 0.5$ fm, in a very good agreement with thermalization time required by hydrodynamic simulations at RHIC.

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